

STRONG CONTINUITY ON HARDY SPACES

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ABSTRACT. We prove the strong continuity of spectral multiplier operators associated with dilations of certain functions on the general Hardy space H_L^1 introduced by Hofmann, Lu, Mitrea, Mitrea, Yan. Our results include the heat and Poisson semigroups as well as the group of imaginary powers.

1. INTRODUCTION

In the theory of semigroups of linear operators on Banach spaces the crucial assumption is that of strong continuity. One often encounters a situation where the semigroup $T_t = e^{-tL}$ is initially defined on $L^2(\Omega)$ and L is a non-negative self-adjoint operator. In this case the spectral theorem immediately gives the strong $L^2(\Omega)$ continuity $\lim_{t \rightarrow 0^+} \|T_t f - f\|_{L^2(\Omega)} = 0$, for $f \in L^2(\Omega)$. Assume additionally that $\{T_t\}_{t>0}$ extends to a locally bounded semigroup on L^p . More precisely, we impose that for each $1 \leq p < \infty$ there exists $t_p > 0$ such that $\|T_t\|_{L^p(\Omega) \rightarrow L^p(\Omega)} \leq C_p$, $t \in [0, t_p]$. Since weak and strong convergence coincide for semigroups of operators (see e.g. [6, Theorem 5.8]), it is straightforward to see that T_t is strongly continuous on all $L^p(\Omega)$, $1 < p < \infty$. Moreover, if we assume that $\{T_t\}_{t>0}$ is contractive on $L^1(\Omega)$, then it is also strongly continuous on $L^1(\Omega)$. Quite often the semigroup $\{T_t\}_{t>0}$ may be also defined on function spaces other than L^p . For instance, if $T_t = e^{t\Delta}$ is the classical heat semigroup on \mathbb{R}^d , then it also acts on the atomic Hardy spaces H_{at}^1 . However, even in this case it is not obvious that the semigroup is strongly continuous on H_{at}^1 .

In this paper we impose that $\{T_t\}_{t>0}$ satisfies the so-called Davies-Gaffney estimates (see (2.3)), and that the underlying space Ω is a space of homogeneous type in the sense of Coifman-Weiss [1]. Under these assumptions, as a corollary of our main result, we prove that e^{-tL} and $e^{-t\sqrt{L}}$ are strongly continuous on the Hardy space H_L^1 . This Hardy space was introduced by Hofmann, Lu, Mitrea, Mitrea, Yan in [8]. Our results are quite general, as there are many operators L satisfying (2.3), e.g. Laplace-Beltrami operators on complete Riemannian manifolds (see e.g. [7, Corollary 12.4]) or Schrödinger operators with non-negative potentials.

The literature on L^p spectral multipliers for operators satisfying Davies-Gaffney estimates is vast. However, as the L^p theory is not discussed in our paper, we do not provide detailed references on this subject. Instead we kindly refer the interested reader to consult e.g. [11] and references therein. There are also results for spectral multipliers on the Hardy space H_L^1 (or more generally H_L^p), see e.g. [3], [4], [5], and [9].

The methods we use are based on [5], in which the authors proved a Hörmander-type multiplier theorem on H_L^1 . The result for semigroups (Corollary 3.2) is a consequence of Theorem 3.1, which treats dilations of more general multipliers than $e^{-\lambda}$. Finally, using Theorem 3.1 we also prove the strong H_L^1 continuity of the group of imaginary powers $\{L^{iu}\}_{u \in \mathbb{R}}$, see Corollary 3.3.

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2. PRELIMINARIES

Let $(\Omega, d(x, y))$ be a metric space equipped with a positive measure μ . We assume that (Ω, d, μ) is a space of homogeneous type in the sense of Coifman-Weiss [1], that is, there exists a constant $C > 0$ such that

$$(2.1) \quad \mu(B_d(x, 2t)) \leq C\mu(B_d(x, t)) \quad \text{for every } x \in \Omega, t > 0,$$

where $B_d(x, t) = \{y \in \Omega : d(x, y) < t\}$. The condition (2.1) implies that there exist constants $C_0 > 0$ and $q > 0$ such that

$$(2.2) \quad \mu(B_d(x, st)) \leq C_0 s^q \mu(B_d(x, t)) \quad \text{for every } x \in \Omega, t > 0, s > 1.$$

In what follows we set n_0 to be the infimum over q in (2.2).

Let $\{e^{-tL}\}_{t>0}$ be a semigroup of linear operators on $L^2(\Omega, d\mu)$ generated by $-L$, where L is a non-negative, self-adjoint operator. We assume additionally that L is injective on its domain. Throughout the paper we impose that $T_t := e^{-tL}$ satisfies Davies-Gaffney estimates, that is,

$$(2.3) \quad |\langle T_t f_1, f_2 \rangle| \leq C \exp\left(-\frac{\text{dist}(U_1, U_2)^2}{ct}\right) \|f_1\|_{L^2(\Omega)} \|f_2\|_{L^2(\Omega)}$$

for every $f_i \in L^2(\Omega)$, $\text{supp } f_i \subset U_i$, $i = 1, 2$, U_i are open subsets of Ω .

Davies-Gaffney estimates are equivalent to the finite speed propagation of the wave equation; the reader interested in this topic is kindly referred to [2]. The finite speed propagation of the wave equation is used in the proof of [5, Lemma 4.8] (our Lemma 2.3), which is an important ingredient in the proof of our main Theorem 3.1.

For $f \in L^2(\Omega)$ we consider the square function $S_h f$ associated with L defined by

$$S_h f(x) = \left(\iint_{\Gamma(x)} |t^2 L T_{t^2} f(y)|^2 \frac{d\mu(y)}{V(x, t)} \frac{dt}{t} \right)^{1/2},$$

where $\Gamma(x) = \{(y, t) \in \Omega \times (0, \infty) : d(x, y) \leq t\}$.

We define the Hardy space $H_L^1 = H_{L, S_h}^1(\Omega)$ as the (abstract) completion of

$$\{f \in L^2(\Omega) : \|S_h f\|_{L^1(\Omega)} < \infty\}$$

in the norm $\|f\|_{H_L^1} = \|S_h f\|_{L^1(\Omega)}$.

It was proved in Hofmann, Lu, Mitrea, Mitrea, Yan [8] that under our assumption (2.3) the space H_L^1 admits the following atomic decomposition.

Let $M \geq 1$, $M \in \mathbb{N}$. A function a is a $(1, 2, M)$ -atom for H_L^1 if there exist a ball $B = B_d(y_0, r) = \{y \in \Omega : d(y, y_0) < r\}$ and a function $b \in \mathcal{D}(L^M)$ such that

$$a = L^M b;$$

$$\text{supp } L^k b \subset B, \quad k = 0, 1, \dots, M;$$

$$\|(r^2 L)^k b\|_{L^2(\Omega)} \leq r^{2M} \mu(B)^{-1/2}, \quad k = 0, 1, \dots, M.$$

We say that $f = \sum_j \lambda_j a_j$ is a $(1, 2, M)$ atomic representation (of f) if $\{\lambda_j\}_{j=0}^\infty \in l^1$, each a_j is a $(1, 2, M)$ atom, and the sum converges in L^2 . Then we set

$$\mathbb{H}_{L, at, M}^1 = \left\{ f : f \text{ has an atomic } (1, 2, M)\text{-representation} \right\},$$

with the norm given by

$$\|f\|_{\mathbb{H}_{L,at,M}^1} = \inf \left\{ \sum_{j=0}^{\infty} |\lambda_j| : f = \sum_{j=0}^{\infty} \lambda_j a_j \text{ is an atomic } (1, 2, M) \text{ representation} \right\}.$$

The space $H_{L,at,M}^1$ is defined as the (abstract) completion of $\mathbb{H}_{L,at,M}^1$.

Theorem 4.14 of [8] asserts that for each $M > n_0/4$ there exists a constant $C > 0$ such that

$$C^{-1} \|f\|_{H_L^1} \leq \|f\|_{H_{L,at,M}^1} \leq C \|f\|_{H_L^1}.$$

In [8] the authors gave also a molecular description of H_L^1 . Fix $\varepsilon > 0$ and $M > n_0/4$, $M \in \mathbb{N}$. We say that a function \tilde{a} is a $(1, 2, M, \varepsilon)$ -molecule associated to L if there exist a function $\tilde{b} \in \mathcal{D}(L^M)$ and a ball $B = B_d(y_0, r)$ such that

$$\tilde{a} = L^M \tilde{b};$$

$$\|(r^2 L)^k \tilde{b}\|_{L^2(U_j B)} \leq r^{2M} 2^{-j\varepsilon} \mu(B(y_0, 2^j r))^{-1/2}$$

for $k = 0, 1, \dots, M$, $j = 0, 1, 2, \dots$, where $U_0 = B$, $U_j(B) = B_d(y_0, 2^j r) \setminus B_d(y_0, 2^{j-1} r)$ for $j \geq 1$. The decomposition $f = \sum_j \lambda_j \tilde{a}_j$ is a $(1, 2, M, \varepsilon)$ molecular representation (of f) if $\{\lambda_j\}_{j=0}^{\infty} \in l^1$, each \tilde{a}_j is a $(1, 2, M, \varepsilon)$ molecule, and the sum converges in L^2 . Then we define

$$\mathbb{H}_{L,mol,M,\varepsilon}^1 = \left\{ f \in L^2(\Omega) : f \text{ has a molecular } (1, 2, M, \varepsilon)\text{-representation} \right\},$$

with the norm given by

$$\|f\|_{\mathbb{H}_{L,mol,M,\varepsilon}^1} = \inf \left\{ \sum_{j=0}^{\infty} |\lambda_j| : f = \sum_{j=0}^{\infty} \lambda_j \tilde{a}_j \text{ is a molecular } (1, 2, M, \varepsilon) \text{ representation} \right\}.$$

The space $H_{L,mol,M,\varepsilon}^1$ is defined as the (abstract) completion of $\mathbb{H}_{L,mol,M,\varepsilon}^1$.

It was proved in [8, Corollary 5.3] that for each $M > n_0/4$ and $\varepsilon > 0$ it holds $\mathbb{H}_{L,at,M}^1 = \mathbb{H}_{L,mol,M,\varepsilon}^1$, with the equivalence of the norms. Moreover, we have $H_L^1 = H_{L,at,M}^1$ and, consequently, $H_L^1 = H_{L,at,M}^1 = H_{L,mol,N,\varepsilon}^1$ for $N, M > n_0/4$.

The following lemma is a slight extension of the observation following the proof of [8, Corollary 5.3].

Lemma 2.1. *Let T be an operator which is bounded on L^2 . Assume that there are $\varepsilon > 0$ and positive integers $M, N > n_0/4$ such that T maps $(1, 2, M)$ atoms uniformly to $(1, 2, N, \varepsilon)$ molecules. More precisely, we impose that there is an $A > 0$ such that $\|T(a)\|_{\mathbb{H}_{L,mol,N,\varepsilon}^1} \leq A \|a\|_{\mathbb{H}_{L,at,M}^1}$ for all $(1, 2, M)$ atoms a . Then T has the unique bounded extension T^{ext} to H_L^1 which satisfies*

$$\|T^{ext} f\|_{H_L^1} \leq C A \|f\|_{H_L^1}.$$

Proof. By density of $\mathbb{H}_{L,at,M}^1$ in $H_{L,at,M}^1 = H_L^1$ it is enough to prove that T is bounded from $\mathbb{H}_{L,at,M}^1$ to $\mathbb{H}_{L,mol,N,\varepsilon}^1$.

Take $f \in \mathbb{H}_{L,at,M}^1$, so that $f = \sum_j \lambda_j a_j$, where a_j are $(1, 2, M)$ atoms, $\{\lambda_j\} \in l^1$, and the sum converges in L^2 . We chose λ_j and a_j in a way that $\sum_j |\lambda_j| \leq 2 \|f\|_{\mathbb{H}_{L,at,M}^1}$. The L^2 boundedness of T implies that $Tf = \sum_j \lambda_j T(a_j)$ is a $(1, 2, N, \varepsilon)$ molecular representation of Tf . Therefore,

$$\|Tf\|_{\mathbb{H}_{L,mol,N,\varepsilon}^1} \leq A \sum_j |\lambda_j| \leq 2A \|f\|_{\mathbb{H}_{L,at,M}^1},$$

and the proof is completed. \square

Let $E := E_{\sqrt{L}}$ be the spectral measure of \sqrt{L} so that

$$Lf = \int_0^\infty \lambda^2 dE(\lambda)f.$$

Then, for a bounded Borel-measurable function $m: [0, \infty) \rightarrow \mathbb{C}$ the spectral multiplier operator $m(\sqrt{L})$ is given on $L^2(\Omega)$ by

$$m(\sqrt{L})f = \int_0^\infty m(\lambda) dE(\lambda)f.$$

Using Lemma 2.1 with $2M$ in place of M and $N = M > n_0/4$ we deduce the following enhancement of [5, Theorem 4.2].

Theorem 2.2. *Assume that m is a bounded function defined on $[0, \infty)$ and such that for some real number $\alpha > (n_0 + 1)/2$ and any nonzero function $\eta \in C_c^\infty(2^{-1}, 2)$ we have*

$$(2.4) \quad \|m\|_{\eta, \alpha} := \sup_{t>0} \|\eta(\cdot)m(t\cdot)\|_{W^{2, \alpha}(\mathbb{R})} < \infty,$$

where $\|F\|_{W^{p, \alpha}(\mathbb{R})} = \|(I - d^2/dx^2)^{\alpha/2} F\|_{L^p(\mathbb{R})}$. Then the operator $m(\sqrt{L})$ extends uniquely to a bounded operator on H_L^1 . Moreover, there exists a constant $C > 0$ such that

$$\|m(\sqrt{L})f\|_{H_L^1} \leq C \|m\|_{\eta, \alpha} \|f\|_{H_L^1}, \quad f \in H_L^1.$$

For the convenience of the reader we also restate Lemma 4.8 of [5].

Lemma 2.3. *Let $\gamma > 1/2$, $\beta > 0$. Then there exists a constant $C > 0$ such that for every even function $F \in W^{2, \gamma + \beta/2}(\mathbb{R})$ and every $g \in L^2(\Omega)$, $\text{supp } g \subset B_d(y_0, r)$, we have*

$$\int_{d(x, y_0) > 2r} |F(2^{-j}\sqrt{L})g(x)|^2 \left(\frac{d(x, y_0)}{r} \right)^\beta d\mu(x) \leq C (r2^j)^{-\beta} \|F\|_{W^{2, \gamma + \beta/2}}^2 \|g\|_{L^2(\Omega)}^2$$

for $j \in \mathbb{Z}$.

Summarizing this section, we may use whichever of the spaces $H_{L, \text{at}, M}^1$ or $H_{L, \text{mol}, M}^1$, $M > n_0/4$, that is convenient.

3. THE RESULTS

We are going to study strong H_L^1 convergence of operators of the form $m(tL)$ as $t \rightarrow 0$. Observe that for the strong L^2 convergence it is enough to assume that m is bounded and continuous at 0. Our first main result is the following theorem.

Theorem 3.1. *Take κ an integer larger than $(n_0 + 1)/2$. Let $m: [0, \infty) \rightarrow \mathbb{C}$ be a continuous function which is C^κ on $(0, \infty)$. Assume that m satisfies the Mihlin condition of order κ , i.e.*

$$(3.1) \quad \sup_{0 \leq j \leq \kappa} \sup_{\lambda > 0} |\lambda^j m^{(j)}(\lambda)| < \infty,$$

and, additionally

$$(3.2) \quad \lim_{\lambda \rightarrow 0^+} \lambda^j m^{(j)}(\lambda) = 0, \quad j = 1, \dots, \kappa.$$

Then, we have the following strong H_L^1 convergence,

$$(3.3) \quad \lim_{t \rightarrow 0^+} m(t\sqrt{L})f = m(0)f, \quad \text{for every } f \in H_L^1.$$

Remark. Straightforward modifications in the proof we present below give a slightly stronger version of the theorem, with the assumption (3.1) replaced by (2.4) for some real number α larger than $(n_0 + 1)/2$.

Before proceeding to the proof let us note the following important corollary.

Corollary 3.2. *Both the heat semigroup e^{-tL} and the Poisson semigroup $e^{-t\sqrt{L}}$ are strongly continuous on H_L^1 .*

Proof of Theorem 3.1. Let M be an integer such that $2M \geq \kappa$. Then $M > n_0/4$. From Theorem 2.2 and the dilation invariance of (2.4) it follows that $m(tL)$ is well-defined and bounded on H_L^1 , uniformly in $t > 0$. Therefore it is enough to prove (3.3) for $f \in \mathbb{H}_{L,at,2M}^1$.

We claim that we can further reduce the proof to demonstrating that

$$(3.4) \quad \lim_{t \rightarrow 0^+} \|m(t\sqrt{L})a - m(0)a\|_{H_L^1} = 0, \quad \text{for } a \text{ being a } (1, 2, 2M)\text{-atom.}$$

Indeed, if (3.4) is true, and $f = \sum_j \lambda_j a_j$ (where $\{\lambda_j\} \in l^1$ and the sum defining f converges also in L^2) then we obtain

$$\|[m(t\sqrt{L}) - m(0)](f)\|_{H_L^1} = \left\| \sum_{j=0}^{\infty} \lambda_j [m(t\sqrt{L}) - m(0)](a_j) \right\|_{H_L^1} \leq \sum_{j=0}^{\infty} |\lambda_j| \|[m(t\sqrt{L}) - m(0)](a_j)\|_{H_L^1}.$$

Now, from Theorem 2.2 it follows that $\|[m(t\sqrt{L}) - m(0)](a_j)\|_{H_L^1}$ is uniformly bounded in t . Therefore, thanks to (3.4) we obtain $\lim_{t \rightarrow 0^+} \|[m(t\sqrt{L}) - m(0)](f)\|_{H_L^1} = 0$, as desired.

To prove (3.4) we will show that there is an $\varepsilon > 0$ such that for every a being a $(1, 2, 2M)$ -atom the function $(m(t\sqrt{L}) - m(0))a$ is a multiple of a $(1, 2, M, \varepsilon)$ molecule and the multiple constant tends to 0 as $t \rightarrow 0$. Note that the rate of convergence may well depend on a for our purposes. There is no loss of generality if we assume that the associated ball B has radius 1, that is $B = B(y_0, 1)$ for certain $y_0 \in \Omega$. This means that $a = L^{2M}b$ where $b \in \mathcal{D}(L^{2M})$, $\text{supp } L^k b \subset B$, and $\|L^k b\|_{L^2} \leq 1$ for $k = 0, 1, \dots, 2M$. Then, denoting $\tilde{b} = [m(t\sqrt{L}) - m(0)]L^M b$, we have $[m(t\sqrt{L}) - m(0)]a = L^M \tilde{b}$. Our task is to study the behavior of L^2 -norms of $L^k \tilde{b} = [m(t\sqrt{L}) - m(0)]L^{k+M}b$, $k = 0, 1, \dots, M$, on the sets $U_j(B)$.

Let $\psi \in C_c^\infty(\frac{1}{2}, 2)$ be such that $\sum_{\ell \in \mathbb{Z}} \psi(2^{-\ell}\lambda) = 1$ for $\lambda > 0$. For $\ell_0 \in \mathbb{Z}$ and $\lambda \in \mathbb{R}$ set $\Psi_{\ell_0}(\lambda) = 1 - \sum_{\ell=\ell_0}^{\infty} \psi(2^{-\ell}|\lambda|)$. We split

$$m(t|\lambda|) - m(0) = \Psi_{\ell_0}(\lambda)(m(t|\lambda|) - m(0)) + \sum_{\ell=\ell_0}^{\infty} \psi(2^{-\ell}|\lambda|)(m(t|\lambda|) - m(0))$$

and for $\lambda \in \mathbb{R}$ put

$$m_{\ell,t}(\lambda) = \psi(2^{-\ell}|\lambda|)(m(t|\lambda|) - m(0)), \quad \tilde{m}_{\ell,t}(\lambda) = m_{\ell,t}(2^\ell \lambda) = \psi(|\lambda|)(m(t2^\ell|\lambda|) - m(0)).$$

Fix $\varepsilon > 0$ and $\gamma > 1/2$ such that $\gamma + \varepsilon + n_0/2 = \alpha$. Set $\beta = n_0 + 2\varepsilon$, so that $\gamma + \beta/2 = \alpha$. Recall that $\text{supp } L^{k+M}b \subset B$ and $m_{\ell,t}(\lambda) = m_{\ell,t}(-\lambda)$. Applying Lemma 2.3 we have

$$\int_{d(x,y_0)>2} |m_{\ell,t}(\sqrt{L})L^{k+M}b(x)|^2 d(x, y_0)^\beta d\mu(x) \leq C2^{-\ell\beta} \|\tilde{m}_{\ell,t}\|_{W^{2,\alpha}} \|L^{k+M}b\|_{L^2}^2,$$

hence, using (3.1) we arrive at

$$\int_{U_j(B)} |m_{\ell,t}(\sqrt{L})L^{k+M}b(x)|^2 d\mu(x) \leq C_\alpha 2^{-\ell\beta} 2^{-j\beta} \|L^{k+M}b\|_{L^2}^2.$$

Therefore

$$(3.5) \quad \left(\int_{U_j(B)} \left| \sum_{\ell > \ell_0} m_{\ell,t}(\sqrt{L}) L^{k+M} b \right|^2 d\mu \right)^{1/2} \leq C_\alpha^{1/2} 2^{-j\beta/2} 2^{-\ell_0\beta/2} \|L^{k+M} b\|_{L^2}.$$

Note that the estimate above does not depend on $t > 0$. For the rest of the proof we fix ℓ_0 large enough.

Denote $n_{\ell_0,t}(\lambda) = \Psi_{\ell_0}(\lambda)(m(t|\lambda|) - m(0))\lambda^{2M}$, $\lambda \in \mathbb{R}$. Clearly, $n_{\ell_0,t}(\lambda) = n_{\ell_0,t}(-\lambda)$. Using Lemma 2.3 we get

$$\int_{d(x,y_0) > 2} |L^k n_{\ell_0,t}(\sqrt{L}) b(x)|^2 d(x, y_0)^\beta d\mu(x) \leq C \|n_{\ell_0,t}\|_{W^{2,\gamma+\beta/2}}^2 \|L^k b\|_{L^2}^2 = C \|n_{\ell_0,t}\|_{W^{2,\alpha}}^2 \|L^k b\|_{L^2}^2,$$

and, consequently,

$$(3.6) \quad \int_{U_j(B)} |L^k n_{\ell_0,t}(\sqrt{L}) b(x)|^2 d\mu(x) \leq C 2^{-j\beta} \|n_{\ell_0,t}\|_{W^{2,\alpha}}^2 \|L^k b\|_{L^2}^2.$$

We claim that $n_{\ell_0,t}(\lambda) = \Psi_{\ell_0}(\lambda)(m(t|\lambda|) - m(0))\lambda^{2M}$ satisfies $\lim_{t \rightarrow 0^+} \|n_{\ell_0,t}\|_{W^{2,\alpha}} = 0$. Indeed

$$\|n_{\ell_0,t}\|_{W^{2,\alpha}} \leq \|n_{\ell_0,t}\|_{W^{2,\kappa}} \approx \|n_{\ell_0,t}\|_{L^2} + \|(n_{\ell_0,t})^{(\kappa)}\|_{L^2} \lesssim C_{l_0} \|(m(t|\lambda|) - m(0))\lambda^{2M}\|_{C^\kappa[0,2^{l_0+1}]},$$

and, because of (3.2), the quantity on the right hand side of the above inequality approaches 0 as $t \rightarrow 0^+$. Summarizing (3.5) and (3.6) we have proved that, for $k = 0, \dots, M$, it holds

$$\begin{aligned} \int_{U_j(B)} |L^{k+M}[m(t\sqrt{L}) - m(0)]b(x)|^2 d\mu(x) &\leq C 2^{-j\beta} (2^{-\ell_0\beta} \|L^{k+M} b\|_{L^2}^2 + \|n_{\ell_0,t}\|_{W^{2,\alpha}}^2 \|L^k b\|_{L^2}^2) \\ &\leq C 2^{-j\beta} (2^{-\ell_0\beta} + \|n_{\ell_0,t}\|_{W^{2,\alpha}}^2) \mu(B(y_0, 1))^{-1} \\ &\leq C 2^{-j\beta} (2^{-\ell_0\beta} + \|n_{\ell_0,t}\|_{W^{2,\alpha}}^2) \frac{\mu(B(y_0, 2^j))}{\mu(B(y_0, 1))} \mu(B(y_0, 2^j))^{-1}. \end{aligned}$$

Using (2.2) with $q = n_0 + \varepsilon$ we obtain

$$\int_{U_j(B)} |L^{k+M}[m(t\sqrt{L}) - m(0)]b(x)|^2 d\mu(x) \leq C 2^{-j\beta} (2^{-\ell_0\beta} + \|n_{\ell_0,t}\|_{W^{2,\alpha}}^2) 2^{jq} \mu(B(y_0, 2^j))^{-1},$$

which is enough for our purpose, since $\beta - q = \varepsilon$, $\gamma + \beta/2 = \alpha$, and $\lim_{t \rightarrow 0^+} \|n_{\ell_0,t}\|_{W^{2,\alpha}}^2 = 0$.

To estimate $L^{k+M}(m(t\sqrt{L}) - m(0))b$ on $2B$, we note that by the spectral theorem,

$$\begin{aligned} \|[m(t\sqrt{L}) - m(0)]L^{k+M} b\|_{L^2(2B)}^2 &\leq \|[m(t\sqrt{L}) - m(0)]L^{k+M} b\|_{L^2(\Omega)}^2 \\ &= \int_0^\infty |m(t\lambda) - m(0)|^2 dE_{L^{k+M}b, L^{k+M}b}(\lambda) \rightarrow 0 \quad \text{as } t \rightarrow 0 \end{aligned}$$

thanks to the Lebesgue dominated convergence theorem and the continuity of m at 0. \square

We finish the paper with showing the strong convergence of the group of imaginary powers. This is achieved by using Theorems 2.2 and 3.1.

Corollary 3.3. *Let $f \in H_L^1$. Then $\lim_{u \rightarrow 0} L^{iu} f = f$, the limit being in H_L^1 .*

Proof. Let $\phi(\lambda)$ be a smooth function on $[0, \infty)$ which is equal to 1 on $[0, 2]$ and vanishes for $\lambda > 4$.

Theorem 2.2 implies

$$(3.7) \quad \sup_{|u| \leq 1} \|L^{iu}\|_{H_L^1 \rightarrow H_L^1} < \infty.$$

Moreover, from Theorem 3.1 it follows that $\lim_{s \rightarrow 0^+} \phi(sL)f = f$, for $f \in H_L^1$ (the limit being in H_L^1). Hence, a density argument together with (3.7) show that it is enough to justify that for each fixed $s > 0$, we have

$$(3.8) \quad \lim_{u \rightarrow 0} (L^{iu} - 1)\phi(sL)f = 0, \quad f \in H_L^1,$$

the limit being understood in H_L^1 . Let M be an integer larger than $(n_0 + 3)/2$. As the linear span of atoms is dense in H_L^1 , in view of (3.7) it suffices to verify (3.8) for f being a fixed $(1, 2, 2M)$ atom. Then $f = L^{2M}b$. Moreover, $a = L^M b$ is a multiple of a $(1, 2, M)$ atom, with a multiple constant that depends on f . Let $m_u(\lambda) = \lambda^{2M}(\lambda^{2iu} - 1)\phi(s\lambda^2)$ and let η be a non-zero smooth function supported in $[1/2, 2]$. A short computation shows that

$$\lim_{u \rightarrow 0} \sup_{t > 0} \|\eta(\cdot)m_u(t \cdot)\|_{W^{2, M-1}} = 0.$$

We also have $(L^{iu} - 1)\phi(sL)f = m_u(\sqrt{L})(a)$ with a being a $(1, 2, M)$ -atom. Since $M - 1 > (n_0 + 1)/2$, using Theorem 2.2 we finish the proof of Corollary 3.3. \square

Remark. Corollary 3.3 seems crucial in extending various results in harmonic analysis based on the group of imaginary powers from the L^p to the H_L^1 setting. For potential applications see e.g. [10] or [12, Remark 3].

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REFERENCES

- [1] Ronald R. Coifman and Guido Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. **83** (1977), no. 4, 569–645.
- [2] Thierry Coulhon and Adam Sikora, *Gaussian heat kernel upper bounds via the Phragmén-Lindelöf theorem*, Proc. Lond. Math. Soc. (3) **96** (2008), no. 2, 507–544. MR 2396848 (2011a:35206)
- [3] Xuan Thinh Duong and Ji Li, *Hardy spaces associated to operators satisfying Davies-Gaffney estimates and bounded holomorphic functional calculus*, J. Funct. Anal. **264** (2013), no. 6, 1409–1437. MR 3017269
- [4] Xuan Thinh Duong and Lixin Yan, *Spectral multipliers for Hardy spaces associated to non-negative self-adjoint operators satisfying Davies-Gaffney estimates*, J. Math. Soc. Japan **63** (2011), no. 1, 295–319.
- [5] Jacek Dziubański and Marcin Preisner, *Remarks on spectral multiplier theorems on Hardy spaces associated with semigroups of operators*, Rev. Un. Mat. Argentina **50** (2009), no. 2, 201–215.
- [6] Klaus-Jochen Engel and Rainer Nagel, *One-parameter semigroups for linear evolution equations*, Graduate Texts in Mathematics, vol. 194, Springer-Verlag, New York, 2000, With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.
- [7] Alexander Grigor'yan, *Heat kernel and analysis on manifolds*, AMS/IP Studies in Advanced Mathematics, vol. 47, American Mathematical Society, Providence, RI; International Press, Boston, MA, 2009.
- [8] Steve Hofmann, Guozhen Lu, Dorina Mitrea, Marius Mitrea, and Lixin Yan, *Hardy spaces associated to non-negative self-adjoint operators satisfying Davies-Gaffney estimates*, Mem. Amer. Math. Soc. **214** (2011), no. 1007, vi+78.
- [9] Peer Christian Kunstmann and Matthias Uhl, *Spectral multiplier theorems of Hörmander type on Hardy and Lebesgue spaces*, J. Operator Theory **73** (2015), no. 1, 27–69. MR 3322756
- [10] Stefano Meda, *A general multiplier theorem*, Proc. Amer. Math. Soc. **110** (1990), no. 3, 639–647. MR 1028046 (91f:42010)

- [11] Adam Sikora, Lixin Yan, and Xiaohua Yao, *Sharp spectral multipliers for operators satisfying generalized Gaussian estimates*, J. Funct. Anal. **266** (2014), no. 1, 368–409. MR 3121735
- [12] Błażej Wróbel, *On the consequences of a Mihlin-Hörmander functional calculus: maximal and square function estimates*, submitted (2015), arXiv:1507.08114.

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